UNSTABLE ANALOGUES OF THE LICHTENBAUM-QUILLEN CONJECTURE

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1. Introduction

This survey is mostly concerned with unstable analogues of the Lichtenbaum-Quillen Conjecture. The Lichtenbaum-Quillen conjecture (now implied by the Voevodsky-Rost Theorem [17]) attempts to describe the algebraic K-theory of rings of integers in number fields in terms of much more accessible "étale models". Suitable versions of the conjecture predict the cohomology of infinite general linear groups of rings of S-integers over suitable number fields; our survey focuses on an unstable version of this form of the conjecture. This survey is not written for experts in algebraic K-theory and should be seen as a computational viewpoint into the mysteries of the cohomology of S-arithmetic groups.

2. The General Problem

The context of our survey can be described as follows: let G be a group scheme, A a ring and ℓ a prime number. In the 1980's Dwyer and Friedlander [6] introduced "étale models" $BG^{et,\ell}(A)$ for the classifying space $BG(A)$ which come with natural maps

$$f = f_{G,A,\ell} : BG(A) \to BG^{et,\ell}(A).$$

These models have turned out to be particularly powerful in the context of algebraic K-theory, i.e., in the case of $G = GL_\infty$, in which the maps $f_{G,A,\ell}$ are supposed to induce an isomorphism in mod-ℓ cohomology for suitable primes ℓ and certain rings A of interest in number theory. In particular, this holds if $A = \mathbb{Z}[1/2]$ and $\ell = 2$ as implied by [16]. In other words, the following question has an affirmative answer in the case that $G = GL_\infty, A = \mathbb{Z}[1/2]$ and $\ell = 2$.

Question $\mathbf{Q}(\boldsymbol{G,A,\ell})$. *Does $f_{G,A,\ell}$ induce an isomorphism in mod-ℓ cohomology?*

The situation turns out to be less favorable in the case of $G = GL_n$ and $G = O_n$. While the answer to question $\mathbf{Q}(\boldsymbol{G,A,\ell})$ is affirmative if $G = GL_n$, $A = \mathbb{Z}[1/2]$ and $\ell = 2$ as long as $n \leq 3$ as proven in [11, 13], it was shown by Dwyer [8] that the answer is negative as soon as $n \geq 32$. In an unpublished work of Lannes and Henn dating from the late 1990's they showed that for $G = O_n, A = \mathbb{Z}[1/2]$ and $\ell = 2$ the answer is yes if and only if $n \leq 14$, and for $G = GL_n, A = \mathbb{Z}[1/2]$ and $\ell = 2$ the answer is no as soon as $n \geq 14$.

In our publications, we study question $\mathbf{Q}(\boldsymbol{G,A,\ell})$ in the case of $G = GL_n$, ℓ an odd regular prime in the sense of number theory and $A = \mathbb{Z}[1/\ell, \zeta_\ell]$, where ζ_ℓ denotes a primitive ℓ-root of unity. It is worthwhile to note that the Voevodsky-Rost Theorem implies that the question in this case has a positive answer if $G = GL_\infty$.

In [1] we study the case of the prime 3. We show by explicit calculation of $H^*(GL_2(\mathbb{Z}[1/3, \zeta_3]); \mathbb{F}_3)$ that the answer is positive if $\ell = 3$ and $n = 2$.

Theorem 1. $H^*(GL_2(\mathbb{Z}[1/3,\zeta_3]);\mathbb{F}_3)\approx\mathbb{F}_3[c_1,c_2]\otimes\Lambda_{\mathbb{F}_3}(e_1,e_2)\otimes\Lambda_{\mathbb{F}_3}(e_1',e_2')$.

Here $\Lambda_{\mathbb{F}_3}(e_1,e_2)$ denotes the exterior algebra generated over $\mathbb{F}_3$ by the elements e_1 and e_2 where the degree of e_j is $2j-1$ and similarly for e_1' and e_2'. The polynomial algebra is generated by c_1 and c_2 where the degree of c_j is $2j$.

This is the first non-trivial positive answer to the question in the case that ℓ is an odd prime. This calculation requires considerable technical strength to combine efficiently existing methods such as the spectral sequence associated to the action of $SL_2(\mathbb{Z}[\zeta_3])$ on a suitable $2-$dimensional contractible complex as well as other spectral sequences converging to the high dimensional cohomology. In the same paper we use the strategy used by Dwyer in [8] to show that the answer to $\mathbf{Q(\boldsymbol{GL_n},\mathbb{Z}[1/3,\boldsymbol{\zeta_3}],3)}$ is negative as soon as $n\geq 27$. More precisely,

Theorem 2. *The following homomorphism*

$$f^*\colon H^*(BGL_n^{et,\ell}(\mathbb{Z}[1/3,\zeta_3]);\mathbb{F}_3)\to H^*(BGL_n(\mathbb{Z}[1/3,\zeta_3]);\mathbb{F}_3)$$

is injective for all $n\geq 0$ but not surjective for $n\geq 27$.

As shown in [10] this disproves at the same time a conjecture of Quillen [14] which predicted that the group cohomology $H^*(GL_n(\mathbb{Z}[1/3,\zeta_3]);\mathbb{F}_3)$ is a free module over $H^*(BGL_n(\mathbb{C});\mathbb{F}_3)$ if the module structure is induced from an embedding of the ring $\mathbb{Z}[1/3,\zeta_3]$ into the complex numbers $\mathbb{C}$. Here $BGL_n(\mathbb{C})$ is the classifying space of the Lie group $GL_n(\mathbb{C})$ and not of the discrete group.

In [2] we show that Dwyer's strategy of [8] can be applied to show that the answer to $\mathbf{Q(\boldsymbol{G},\boldsymbol{A},\boldsymbol{\ell})}$ is negative if $G=GL_n$ for n sufficiently large if ℓ is an odd regular prime and if $A=\mathbb{Z}[1/\ell,\zeta_\ell]$.

Theorem 3. *If ℓ is an odd regular prime, then the map*

$$f=f_n\colon BGL_n(\mathbb{Z}[1/\ell,\zeta_\ell])\to BGL_n^{et,\ell}(\mathbb{Z}[1/\ell,\zeta_\ell])$$

does not induce an isomorphism on mod-ℓ cohomology for n sufficiently large.

The proof requires a careful analysis of the étale model and uses (as in [8]) deep results from the homotopy theory of classifying spaces, which ultimately depend on the solution of the Sullivan conjecture.

Remark 4. The map f_n in Theorem 3 induces an injection on mod ℓ cohomology for all $n\geq 0$ by [7] but not a surjection for $n\geq N_\ell$ where N_ℓ depends on ℓ.

In [3] the prime ℓ and the ring A are as in [2]. The focus is on analyzing the image I_n of $H^*(BGL_n(A);\mathbb{F}_\ell)$ in $H^*(BD_n(A);\mathbb{F}_\ell)$ with respect to the restriction homomorphism res_n associated to the inclusion of the subgroup $D_n(A)$ of diagonal matrices into $GL_n(A)$. This is compared with the image M_n of the composition of $res_n\circ f_n^*$. It is clear that $M_n\subset I_n$. Furthermore, the étale model can be explicitly described and M_n can be explicitly calculated as in [7].

Theorem 5. *If ℓ is an odd regular prime, then $res_n\circ f_n^*$ induces an isomorphism*

$$H^*(BGL_n^{et,\ell}(\mathbb{Z}[1/\ell,\zeta_l]);\mathbb{F}_\ell)\approx M_n=\mathbb{F}_\ell[c_1,c_2,\dots,c_r]\otimes\Lambda_{\mathbb{F}_\ell}(e_{i,1},e_{i,2},\dots,e_{i,r})^{\otimes_{i=1}^r}$$

where $r=\frac{1}{2}(\ell+1)$, the generator c_j has degree $2j$ and $e_{i,j}$ has degree $2j-1$.

The main result of the paper says that $M_2=I_2$ implies $M_n=I_n$ for all $n\geq 2$; in other words, if one can verify for $n=2$ that the image of the restriction is not larger than the image of the étale model, then this holds true for every $n\geq 2$.

Theorem 6. *If ℓ is an odd regular prime, then the images of the maps $res_n \circ f_n^*$ and res_n agree i.e., $M_n = I_n$ for all $n \geq 0$, if and only if $M_2 = I_2$.*

The proof of this statement relies on the solution to a problem in invariant theory involving the canonical (signed) permutation action of the symmetric group Σ_n on graded rings of the form $(\mathbb{F}_\ell[x] \otimes \Lambda_{\mathbb{F}_\ell}(y_1, \ldots, y_r))^{\otimes n}$.

3. Homological Symbols

Unfortunately the verification of $M_2 = I_2$ is extremely hard to verify as soon as the prime ℓ is bigger than 3. In order to pin down where the difficulties lie, in [4] we concentrate our attention on the image of the canonical homomorphism

$$H_i(BD_j(\mathbb{Z}[1/\ell, \zeta_\ell]); \mathbb{F}_\ell) \to H_i(BGL_j(\mathbb{Z}[1/\ell, \zeta_\ell]); \mathbb{F}_\ell)$$

induced by the inclusion of the subgroup of diagonal matrices $D_j \subset GL_j$. As i and j vary, one gets a homomorphism of bi-graded algebras.

Definition 7. We call the image of this homomorphism the *algebra of homological symbols* and denote it by $KH_{**}(\mathbb{Z}[\zeta_\ell, 1/\ell])$.

We proposed to compare this to an *algebra of étale homological symbols* similarly defined in terms of the étale unstable models for $BGL_j(\mathbb{Z}[\zeta_\ell, 1/\ell])$ and we denote that algebra by $KH^{et}_{**}(\mathbb{Z}[\zeta_\ell, 1/\ell])$. We had explicitly determined the algebra KH^{et}_{**} in terms of generators and relations as follows.

The mod ℓ group homology of GL_1 is the homology of the bar resolution (B_*, ∂) where B_i is the vector space over $\mathbb{F}_\ell$ generated by the symbols $[x_1|\ldots|x_i]$ with $x_i \in GL_1 \setminus \{1\}$ and ∂ is the boundary homomorphism

$$\partial[x_1|\ldots|x_i] = [x_2|\ldots|x_i] + \sum_{j=1}^{i-1}(-1)^j[x_1|\ldots|x_jx_{j+1}|\ldots|x_i] + (-1)^i[x_1|\ldots|x_{i-1}]$$

where we delete all the terms with $x_jx_{j+1} = 1$. We define the shuffle product of two symbols by the formula

$$[x_1|\ldots|x_i] \wedge [x_{i+1}|\ldots|x_{i+s}] = \sum_\sigma (-1)^\sigma [x_{\sigma(1)}|\ldots|x_{\sigma(i+s)}]$$

where σ runs over all the permutations of $\{1, \ldots, i+s\}$ that shuffle $\{1, \ldots, i\}$ with $\{i+1, \ldots, i+s\}$ and $(-1)^\sigma$ denotes the signature of σ.

By [18], GL_1 is the Abelian group generated by the set of cyclotomic units

$$S = \{-\zeta_\ell, 1-\zeta_\ell, 1-\zeta_\ell^2, \ldots, 1-\zeta_\ell^r\}$$

subject to one relation $(-\zeta_\ell)^{2\ell} = 1$. Here we note that $\ell = 2r+1$. By a Künneth isomorphism, the bi-graded algebra $H_i(BD_j(\mathbb{Z}[\zeta_\ell, 1/\ell]); \mathbb{F}_\ell)$ is the bi-graded tensor algebra generated by the cycles

$$[\zeta_\ell]^{(s)} \wedge \langle v_1, \ldots, v_i\rangle := \sum_{i_1,\ldots,i_s=1}^{\ell-1} [\zeta_\ell^{i_1}|\zeta_\ell|\ldots|\zeta_\ell^{i_s}|\zeta_\ell] \wedge [v_1] \wedge \cdots \wedge [v_i] \tag{1}$$

where $\{v_1, \ldots, v_i\} \subset S$ is any subset of cyclotomic units, s is any nonnegative integer, and the bi-degree of this cycle is $(i+2s, 1)$.

Definition 8. A cycle of the form (1) is an *étale obstruction cycle* if $i - s$ is > 0 and even and it is an *odd cycle* if $i + s$ is odd.

Theorem 9. *If ℓ is an odd regular prime, then KH^{et}_{**} is the bi-graded algebra generated by cycles of the form (1) and subject to the relations*

$$\rho_*(t_*(z) \otimes z') = 0,\ t(u) = u^{-1} \times u,\ \rho(u \times v \times w) = uw \times vw$$

where z runs over the étale obstruction cycles and z' over the odd cycles.

Here t_* and ρ_* are induced at the level of cycles by the group homomorphisms $t\colon GL_1 \to GL_1^{\times 2}$ and $\rho\colon GL_1^{\times 3} \to GL_1^{\times 2}$ defined in the theorem. Moreover there is a canonical surjection of bi-graded algebras

$$KH_{**}(\mathbb{Z}[\zeta_\ell, 1/\ell]) \to KH^{et}_{**}(\mathbb{Z}[\zeta_\ell, 1/\ell])$$

and we conjecture

Conjecture 1. *This homomorphism is an isomorphism for ℓ a regular odd prime.*

We verify this conjecture in the case of the prime $\ell = 3$ and indicate a program how one might attack the case of larger primes. The case of each prime would have to be dealt with separately and the computational complexity would grow rapidly with the prime. As a consequence of Theorem 9 we have the following result [3]:

Theorem 10. *The Conjecture 1 is true if and only if $t_*(z)$ is null-homologous in $H_*(SL_2(\mathbb{Z}[1/\ell, \zeta_\ell]); \mathbb{F}_\ell)$ for all étale obstruction cycles z.*

Remark 11. The Conjecture 1 is equivalent to $M_2 = I_2$ as in Theorem 6. Observe that the image of the homomorphism t is contained in the diagonal subgroup of SL_2 and there are finitely many étale obstruction cycles to check.

4. Some algorithms

These verifications are usually not accessible without the algorithms developed by the second author in [15]. Namely, given G a finitely-presented group and k a finite field, the paper exploits a formula due to Hopf to algorithmically give upper bounds on the dimension d of the second homology group of G as a vector space over k.

In particular, if $G = F/R$ is a finite presentation, we have the short exact sequence

$$1 \to [F, R] \to R \cap [F, F] \to H_2(G) \to 1.$$

Through a series of short exact sequences, we show that the dimension of $H_2(G, k)$ is less than or equal to $a+b-c+e$, where a is the dimension of $\mathrm{Tor}(H_1(G), k)$, ℓ^b and ℓ^c are the orders of the ℓ-primary subgroups of $F/R[F, F]$ and $F/R^\ell[F, F]$ respectively, e is the vector space dimension of $k \otimes R/[F, R]$, and ℓ is the characteristic of k. All of these numbers except for e can be calculated by algorithms given in [15]. In general, e can only be estimated from above using the following series of algorithms given in pseudocode.

Algorithm. ReduceWord(F, R, Z, R', ℓ)

Require: *Free Group F, Relators R, Test Word z, Sublist R' of R, Prime ℓ*
Ensure: *Reduced word of z in $F/[F, R]R^\ell R'$*

1: $G := F/[F, R]R^\ell R'$
2: RG :=*Rewriting system for G*
3: x :=*Reduced word of (z) in the rewriting system RG*
4: ***return*** x

This algorithm attempts to reduce a given test word in a finitely-presented group using a rewriting system. We utilized the KBMAG package implemented on GAP for this purpose.

Algorithm. FindBasis(F, R, ℓ, R')

Require: *Free Group F, Relators R, Prime ℓ, Sublist R' of R*
Ensure: *Size of a generating set for $[F,R]R^\ell R'/[F,R]R^\ell$*

```
X := R'
for x ∈ X do
    x' := ReduceWord(F, R, x, Difference(X, [x]), ℓ) {Difference(A, B) is the complement of B in A}
    if x' = identity then
        X := Difference(X, [x])
    end if
end for
return Size(X)
```

The algorithm tries to determine the linear independence of elements x in a generating set R' with respect to $R' - \{x\}$ in $[F,R]R^\ell R'/[F,R]R^\ell$. If x is determined to be dependent on $R' - \{x\}$, it is removed from R'. The end result will be a list of generators which are, at least potentially, linearly independent.

Algorithm. SecondHomologyCoefficients(F, R, ℓ, R')

Require: *Free Group F, Relators R, Prime ℓ, Sublist R' of R generating $R/[F,R]R^\ell$*
Ensure: *An integer d such that dim $(H_2(G;k)) \leq d$*

```
a := Tor(F, R, ℓ)
b := PrimePrimaryRank(F, R[F, F], ℓ)
c := PrimePrimaryRank(F, R^p[F, F], ℓ)
e := FindBasis(F, R, ℓ, R')
d := a + b − c + e
return d
```

Based on finite presentations for groups $G = SL_2(\mathbb{Z}[1/\ell, \zeta_\ell])$ in [4] and our algorithms in [15] we prove that $d = 0$ for $\ell = 3, 5$ and $d \leq 6$ for $\ell = 7$.

Theorem 12. *The homology group $H_2(SL_2(\mathbb{Z}[1/\ell, \zeta_\ell]); \mathbb{F}_\ell) = 0$ if $\ell = 3, 5$ and it is at most 6-dimensional as a vector space over $\mathbb{F}_\ell$ if $\ell = 7$.*

To prove this theorem we define a group SE_2 for $\ell = 2r+1$ by a finite presentation

$$1 \to R \to F \to SE_2 \to 1$$

where F is the free group given by the generators $z, u_1, ..., u_r, a, b$ and R is the normal subgroup given by the following relations

$$z^\ell = [z, u_i] = [u_i, u_j] = a^4 = [a^2, z] = [a^2, u_i] = [b_s, b_t] = c(I)^3 = 1$$
$$a = zaz = u_i a u_i, b^3 = a^2 = b_0 b_1 ... b_{2r}, b_t^\ell = w^{-1} b_t^{(-1)^r} w,$$
$$ba^2 = u_i b z^{-ri} b^{-1} b_0^{-1} z^{ri} b z^{-i} u_i$$

where $i, j \in \{1, 2, ..., r\}$, $s, t \in \{0, 1, 2, ..., 2r\}$, and $I \subset \{1, 2, ..., r\}$ nonempty. The notations are as follows

$$b_t := z^{rt} b z^{rt} a, w := z^c u_1 u_2 ... u_r, c(I) := (\prod_{t=0}^{2r} b_t^{c_t(I)}) a^{-1} \prod_{i \in I} u_i$$

where $c, c_t(I) \in \mathbb{Z}$ with $c \geq 0$ minimal such that

$$2c \equiv r^2 + \frac{r(r+1)}{2} \mod \ell, \prod_{i \in I}(1 - \zeta_\ell^i) = \sum_{t=0}^{2r} c_t(I)\zeta_\ell^t.$$

The conventions are $b_t = b_s$ if $s \equiv t \mod \ell$ and $[x, y] = xyx^{-1}y^{-1}$. Then there is a group homomorphism $SE_2 \to SL_2$ such that Conjecture 1 is true if the cycles

$$[z]^{(s)} \wedge [e_1] \wedge ... \wedge [e_i]$$

are null-homologous in $H_{3s+2j}(SE_2; \mathbb{F}_\ell)$ for each pair of nonnegative integers (s, j) and subset $\{e_1, ..., e_i\} \subset \{z, u_1, ..., u_r\}$ with $i = s + 2j$ and $j > 0$. Our algorithms apply in the case $(s, j) = (0, 1)$ as follows

Lemma 13. *The cycle* $[e_1] \wedge [e_2]$ *is null-homologous in* $H_2(SE_2; \mathbb{F}_\ell)$ *if and only if* $[e_1, e_2] \in [F, R]R^\ell$.

Combining Theorems 10 and 12 we conclude that Conjecture 1 is true if $\ell = 3$. For $\ell \geq 5$ our algorithms need to be improved to handle higher computational complexity for our conjecture. Nevertheless, there are a couple of new applications we consider in our work in progress [5].

We are constructing, as a by product of the calculations relevant for this paper, a database for low dimensional group homology of linear groups. This will be extended to other finitely-presented groups of interest in number theory and computational topology. An initial set of these calculations is given in Table 1. While the results in the table are not new, they were previously found by a variety of methods, many of which are not computational. A $\leq$ indicates that only an upper bound was found.

	$H_2(-;\mathbb{F}_2)$	$H_2(-;\mathbb{F}_3)$	$H_2(-;\mathbb{F}_5)$	$H_2(-;\mathbb{F}_7)$
$GL_2(\mathbb{Z})$	≤ 4	≤ 2	≤ 2	≤ 2
$SL_2(\mathbb{Z})$	≤ 2	≤ 2	≤ 1	≤ 1
$SL_2(\mathbb{Z}_2)$	1	0	0	0
$SL_2(\mathbb{Z}_3)$	0	1	0	0
$SL_2(\mathbb{Z}_5)$	0	0	1	0
$SL_2(\mathbb{Z}[i])$	1	0	0	0
$SL_2(\mathbb{Z}[\omega]), \omega^3 = -1$	≤ 1	≤ 2	≤ 1	≤ 1
$SL_2(\mathbb{Z}[\sqrt{-5}])$	≤ 3	≤ 3	0	0
$PSL_2(\mathbb{Z})$	≤ 1	≤ 1	0	0

Table 1. Dimensions of Second Homology Groups

Also in [5] we explain how the algorithms in [15] can be adapted to find generators of H_2. For example, results in [4] show that $SL_2(\mathbb{Z}[1/7, \zeta_7])$ has presentation with generators $\{z, u_1, u_2, u_3, a, b, b_0, b_1, b_2, b_3, b_4, b_5, b_6, w\}$ and a set of 64 relators.

Theorem 14. *As a vector space over* $\mathbb{F}_7$*,* $H_2(SL_2(\mathbb{Z}[1/7, \zeta_7]); \mathbb{F}_7)$ *is generated by the following set of relators*

$$\begin{aligned}&\{zaza^{-1},\\ &u_1u_2u_1^{-1}u_2^{-1},\\ &u_1au_1a^{-1},\\ &b_1a^{-1}b_1a^{-1}b_1a,\\ &u_2b_1z^{-2}b_1zb_1^{-2}a^{-1}u_2z^{-1}a^{-1}b_1^{-1},\\ &u_3b_1z^2b_1^{-2}z^2b_1a^{-1}z^{-2}u_3z^{-1}a^{-1}b_1^{-1}\}.\end{aligned}$$

To summarize, significant contributions were made, which help clarify the limitation of étale models for finite general linear groups over rings of S-integers of number fields, but much remains to be done in low dimensional group homology.

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